The Spectrum of Independence

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Independence Number

A family $\mathscr{A} \subseteq [\omega]^\omega$ is said to be independent for any two non-empty finite disjoint subfamilies $\mathscr{A}_0$ and $\mathscr{A}_1$ the set

$$\bigcap \mathscr{A}_0 \setminus \bigcup \mathscr{A}_1$$

is infinite. It is a maximal independent family if it is maximal under inclusion and

$$\mathfrak{i} = \min\{|\mathscr{A}| : \mathscr{A} \text{is a m.i.f.}\}$$

Boolean combinations

- Functions $h : \mathscr{A} \to \{0,1\}$ where $|\mathrm{dom}(\mathscr{A})| < \omega$ and $\mathscr{A}^h = \bigcap\{A : A \in h^{-1}(0)\} \cap \bigcap\{\omega \setminus A : A \in h^{-1}(1)\}$.
- $\mathrm{FF}(\mathscr{A}) = \{h : \mathscr{A} \to \{0,1\} \mid |\mathrm{dom}\, h| < \omega\}$.

$\{\mathscr{A}^h : h \in \mathrm{FF}(\mathscr{A})\}$ is the collection of all Boolean combinations of $\mathscr{A}$.

Countable independent families are not maximal

Let $\mathscr{A}$ be a countable independent family and let $\{h_n\}_{n\in\omega}$ be an enumeration of $\mathrm{FF}(\mathscr{A})$ so that each element appears cofinally often. Inductively define $\{a_{2n}, a_{2n+1}\}_{n\in\omega}$ so that

$$a_{2n}, a_{2n+1} \text{belong to } \mathscr{A}^{h_n} \backslash \{a_{2k}, a_{2k+1}\}_{k<n}.$$

Then $A = \{a_{2n}\}_{n\in\omega}$ is independent over $\mathscr{A}$.

Fichtenholz-Kantorovich

Let $C = [\mathbb{Q}]^{<\omega}$ and for $r \in \mathbb{R}$ let

$$A_r = \{a \in C : a \cap (-\infty, r] \text{ is even}\}.$$

Then whenever S, T are finite disjoint sets of reals, the set

$$\bigcap_{r \in S} A_r \cap (C \setminus \bigcup_{r \in T} A_r)$$

is infinite. Thus, there is always a m.i.f. of size $\mathfrak{c}$.

$\mathfrak{r} \leq \mathfrak{i}$

Let $\mathscr{A}$ be a m.i.f. and $X \in [\omega]^\omega \backslash \mathscr{A}$. By maximality of $\mathscr{A}$, $\exists h \in \mathrm{FF}(\mathscr{A})$ such that either $\mathscr{A}^h \cap X$ or $\mathscr{A}^h \backslash X$ is finite. Thus $\mathscr{A}^h$ is not split by X.

$\mathfrak{d} \leq \mathfrak{i}$

If $\mathscr{D} \subseteq {}^\omega\omega$ is such that for each $h \in {}^\omega\omega$ there is $g \in \mathscr{D}$ such that $h(n) \leq g(n)$ for all but finitely many n, then $|\mathscr{D}| \leq \mathfrak{i}$.

$\mathfrak{i}$ vs. $\mathfrak{u}$

In the Miller model $\mathfrak{u} < \mathfrak{i}$, while Shelah devised a special ${}^\omega\omega$-bounding poset the countable support iteration of which produces a model of $\mathfrak{i} = \aleph_1 < \mathfrak{u} = \aleph_2$.

$\mathfrak{a}$ vs. $\mathfrak{u}$

In the Cohen model $\mathfrak{a} < \mathfrak{u}$, while assuming the existence of a measurable one can show the consistency of $\mathfrak{u} < \mathfrak{a}$. The use of a measurable has been eliminated by Guzman and Kalajdzievski.

$\mathfrak{a}$ vs $\mathfrak{i}$

In the Cohen model $\mathfrak{a} < \mathfrak{i} = \mathfrak{c}$.

Question:

Is it consistent that $\mathfrak{i} < \mathfrak{a}$?

... and once again Maximality

$\forall X \in [\omega]^\omega \setminus \mathscr{A} \exists h \in \mathrm{FF}(\mathscr{A})$ such that $\mathscr{A}^h \cap X$ or $\mathscr{A}^h \setminus X$ is finite.

Dense maximality

Let $\mathscr{A}$ be an independent family. Then $\mathscr{A}$ is said to be densely maximal if for each $X \in [\omega]^\omega \setminus \mathscr{A}$ and every $h \in \mathrm{FF}(\mathscr{A})$ there is $h' \in \mathrm{FF}(\mathscr{A})$ such that $h' \supseteq h$ and $\mathscr{A}^{h'} \cap X$ or $\mathscr{A}^{h'} \setminus X$ is finite.

Density filter

Let $\mathscr{A}$ be an independent family. Then

$$\text{fil}(\mathscr{A}) = \{Y \in [\omega]^\omega : \forall h \in \text{FF}(\mathscr{A}) \exists h' \in \text{FF}(\mathscr{A}) \text{ s.t. } h' \supseteq h \text{ and } \mathscr{A}^{h'} \subseteq Y\}$$

is referred to as the density filter of $\mathscr{A}$.

Definition: Ramsey filter

A p-filter $\mathscr{F}$ is said to be Ramsey if for every partition $\mathscr{E} = \{E_n\}_{n\in\omega}$ of ω into finite sets such that $\omega \backslash E_n \in \mathscr{F}$ for each n, there is a set $\{k_n\}_{n\in\omega}$ in $\mathscr{F}$ such that $k_n \in E_n$ for each n.

Definition: Selective independence

A densely maximal independent family $\mathscr{A}$ is said to be selective if $\mathrm{fil}(\mathscr{A})$ is Ramsey.

Theorem (Shelah)

- Selective independent families exists under *CH*.
- They are indestructible by a countable support iterations and countable support products of Sacks forcing.

Corollary

It is consistent that $\mathfrak{i} < \mathfrak{c}$.

Definition

Let $\mathbb{P}$ be the partial order

- of all pairs $(\mathscr{A}, A)$ where $\mathscr{A}$ is a countable independent family and $A \in [\omega]^\omega$ such that for all $h \in \mathrm{FF}(\mathscr{A})$ the set $\mathscr{A}^h \cap A$ is infinite;
- with extension relation defined as follows

$$(\mathscr{B}, B) \leq (\mathscr{A}, A) \text{ iff } \mathscr{B} \supseteq \mathscr{A} \text{ and } B \subseteq^* A.$$

Lemma (CH)

The partial order $\mathbb{P}$ is countably closed and $\aleph_2$-cc. Moreover, if G is $\mathbb{P}$-generic, then $\mathscr{A}_G = \bigcup\{\mathscr{A} : \exists A(\mathscr{A}, A) \in G\}$ is a selective independent family.

More precisely

- $\mathscr{A}_G$ is densely maximal;
- $\text{fil}(\mathscr{A}_G)$ is generated by $\{A : \exists \mathscr{A}(\mathscr{A}, A) \in G\} \cup \textbf{Fr}$;
- $\text{fil}(\mathscr{A})$ is Ramsey.

Definition: Spectrum of Independence

$\mathfrak{sp}(\mathfrak{i}) = \{|\mathscr{A}| : \mathscr{A}$ is a max. ind. family$\}$

Theorem (F., Shelah)

Assume *CH*. Let κ be a regular uncountable cardinal. Then

$$V^{\mathbb{S}_\kappa} \vDash \mathfrak{sp}(\mathfrak{i}) = \{\aleph_1, \kappa\}.$$

$\mathscr{A}$-diagonalization filters (F., Shelah)

Let $\mathscr{A}$ be an independent family. A filter $\mathscr{U}$ is said to be an $\mathscr{A}$-diagonalization filter if

$$\forall F \in \mathscr{U} \forall h \in \mathrm{FF}(\mathscr{A})(|F \cap \mathscr{A}^h| = \omega)$$

and is maximal with respect to the above property.

Lemma (F., Shelah)

If $\mathscr{U}$ is a $\mathscr{A}$-diagonalization filter and G is $\mathbb{M}(\mathscr{U})$-generic and $x_G = \bigcup\{s : \exists F(s, F) \in G\}$, then:

1. $\mathscr{A} \cup \{x_G\}$ is independent
2. If $y \in ([\omega]^\omega \backslash \mathscr{A}) \cap V$ is such that $\mathscr{A} \cup \{y\}$ is independent, then $\mathscr{A} \cup \{x_G, y\}$ is not independent.

Definition

We say that y diagonalizes $\mathscr{A}$ over V_0 (in V_1) iff

1. V_1 extends V_0, $(\mathscr{A}$ is independent$)^{V_0}$
2. whenever $x \in ([\omega]^{\aleph_0})^{V_0} \setminus \mathscr{A}$ such that $V_0 \models \mathscr{A} \cup \{x\}$ is independent, then $V_1 \models \mathscr{A} \cup \{x, y\}$ is not independent.

Corollary

If $\mathscr{U}$ an $\mathscr{A}$-diagonalization filter and G is a $\mathbb{M}(\mathscr{U})$-generic, then $\sigma_G = \bigcup\{s : \exists A(s, A) \in G\}$ diagonalizes $\mathscr{A}$ over the ground model.

Corollary

Let κ be a regular uncountable cardinal. Then consistently

$$\aleph_1 < \mathfrak{i} = \kappa < \mathfrak{c}.$$

Proof:

Let $\lambda > \kappa$ be the desired size of the continuum. The ordinal product $\gamma^* = \lambda \cdot \kappa$ contains an unbounded subset $\mathscr{I}$ of cardinality κ. Consider a finite support iteration of length γ^* such that along $\mathscr{I}$ we

- recursively generate a max. independent family of cardinality κ,
- as well as a scale of length κ,

and along $\gamma^* \backslash \mathscr{I}$, we add Cohen reals. Then in the final generic extension

$$\aleph_1 < \mathfrak{d} = \kappa \leq \mathfrak{i} \leq \kappa < \mathfrak{c} = \lambda.$$

□

Question:

Can we adjoin via forcing a max. independent family of cardinality $\aleph_\omega$?

Theorem (F., Shelah)

Assume *GCH*. Let $\kappa_1 < \cdots < \kappa_n$ be regular uncountable cardinals. There is a ccc generic extension in which $\{\kappa_i\}_{i=1}^n \subseteq \mathfrak{sp}(\mathfrak{i})$.

Proof:

Consider a finite support iteration of length γ^*, where γ^* is the ordinal product $\kappa_n \cdot \kappa_{n-1} \cdots \kappa_1$ and elaborate on the previous idea. □

Ultrapowers

Let κ a measurable and let $\mathscr{D} \subseteq \mathscr{P}(\kappa)$ be a κ-complete ultrafilter. Let $\mathbb{P}$ be a p.o. Then $\mathbb{P}^\kappa/\mathscr{D}$ consists of all equivalence classes

$$[f] = \{g \in {}^\kappa\mathbb{P} : \{\alpha \in \kappa : f(\alpha) = g(\alpha)\} \in \mathscr{D}\}$$

and is supplied with the p.o. relation $[f] \leq [q]$ iff

$$\{\alpha \in \kappa : f(\alpha) \leq_{\mathbb{P}} g(\alpha)\} \in \mathscr{D}.$$

We can identify each $p \in \mathbb{P}$ with $[p] = [f_p]$, where $f_p(\alpha) = p$ for each $\alpha \in \kappa$ and so we can assume $\mathbb{P} \subseteq \mathbb{P}^\kappa/\mathscr{D}$.

Lemma

1. The poset $\mathbb{P}$ is a complete suborder of $\mathbb{P}^\kappa/D$ if and only if $\mathbb{P}$ is κ-cc. Thus, if $\mathbb{P}$ is ccc, then $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathscr{D}$.
2. If $\mathbb{P}$ has the countable chain condition, then so does $\mathbb{P}^\kappa/\mathscr{D}$.

Lemma

Let $\mathscr{A}$ be a $\mathbb{P}$-name for an independent family of cardinality $\geq \kappa$. Then

$$\Vdash_{\mathbb{P}^\kappa/\mathscr{D}} \mathscr{A} \text{ is not maximal.}$$

Theorem (F., Shelah, 2018)

Let $\kappa_1 < \kappa_2 < \cdots < \kappa_n$ be measurable witnessed by κ_i-complete ultrafilters $\mathscr{D}_i \subseteq \mathscr{P}(\kappa_i)$. There is a ccc generic extension in which

$$\{\kappa_i\}_{i=1}^n = \mathfrak{sp}(\mathfrak{i}).$$

Proof/Idea:

Let $\gamma^* = \kappa_n \cdot \kappa_{n-1} \cdots \kappa_1$ and for each $j \in \{1, \cdots, k\}$ fix an unbounded subset $\mathscr{I}_j$ in γ^*. Along each $\mathscr{I}_j$

- iteratively generate a max. ind. family of cardinality κ_j
- and for unboundedly many $\alpha \in \mathscr{I}_j$ take the ultrapower $\mathbb{P}_\alpha^{\kappa_j} / \mathscr{D}_j$.

□

Do we need a measurable?

Lemma

Let $\mathscr{A}$ be an independent family and let $\mathscr{U}$ be a diagonalization filter for $\mathscr{A}$. Let $n \in \omega$ and for each $i \in n$ let $\mathscr{U}_i = \mathscr{U}$. Moreover let $G = \prod_{i\in n} G_i$ be a $\mathbb{P} = \prod_{i\in n}\mathbb{M}(\mathscr{U}_i)$-generic filter. Then in $V[G]$:

1. $\mathscr{A} \cup \{x_i\}_{i\in n}$ is independent.
2. For all $y \in (V\backslash\mathscr{A}) \cap [\omega]^\omega$ such that $\mathscr{A} \cup \{y\}$ is independent and each $i \in n$, the family $\mathscr{A} \cup \{y, x_i\}$ is not independent.

Claim (GCH)

- Given an arbitrary uncountable cardinal θ, there is a ccc poset, which adjoins a max. independent family of cardinality θ.
- In particular, there is a ccc poset adjoining a maximal independent family of cardinality $\aleph_\omega$.

Definition

Fix $\sigma \leq \theta \leq \lambda$, where:

- σ is regular uncountable (the intended value of $\mathfrak{i}$),
- λ is of uncountable cofinality (the intended value of $\mathfrak{c}$).
- Let $S \subseteq \theta^{<\sigma}$ be a well-prunded θ-splitting tree of height σ.
- For each $\alpha < \sigma$, let S_α be the α-th level of S.

Recursively define a finite support iteration

$$\mathbb{P}_S = \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \sigma, \beta < \sigma \rangle$$

of length σ as follows:

- Let $\mathbb{P}_0 = \{\emptyset\}$, $\dot{\mathbb{Q}}_0$ be a $\mathbb{P}_0$-name for the trivial poset.
- Let $\mathscr{A}_0 = \emptyset$ and let $\mathscr{U}_0$ be an arbitrary ultrafilter extending the Frechét filter. For each $\eta \in S_1 = \mathrm{succ}_S(\emptyset)$, let $\mathscr{U}_\eta = \mathscr{U}_0$ and let

$$\mathbb{Q}_1 = \prod_{\eta \in S_1} \mathbb{M}(\mathscr{U}_\eta)$$

  with finite supports.
- In $V^{\mathbb{P}_1 * \dot{\mathbb{Q}}_1}$ for each $\eta \in S_1$ let a_η be the $\mathbb{M}(\mathscr{U}_\eta)$-generic real.
- Suppose $\alpha \geq 2$ and in $V^{\mathbb{P}_\alpha}$ for all $\eta \in S_\alpha$,

$$\mathscr{A}_\eta = \{a_\nu : \nu \in \mathrm{succ}_S(\eta \upharpoonright \xi), \xi < \alpha\}$$

  is independent. For each $\eta \in S_\alpha$, let $\mathscr{U}_\eta$ be a $\mathscr{A}_\eta$-diagonalization filter and let $\mathbb{Q}_\alpha = \prod_{\eta \in S_\alpha} \mathbb{M}(\mathscr{U}_\eta)$ with finite supports.
- In $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha}$ for each $\eta \in S_\alpha$ let a_η be the $\mathbb{M}(\mathscr{U}_\eta)$-generic real.

Lemma

In $V^{\mathbb{P}_S}$ for each branch $\eta \in [S]$ the family

$$\mathscr{A}_\eta = \{a_\nu : \nu \in \mathrm{succ}(\eta \restriction \xi), \xi < \alpha\}$$

is a maximal independent family of cardinality θ.

Proof:

Maximality follows from the diagonalization properties and the fact that the length of the iteration is of uncountable cofinality. □

Theorem (F., Shelah, 2020)

Assume GCH. Let σ be a regular uncountable cardinal, λ a cardinal of uncountable cofinality such that $\sigma \leq \lambda$. Let

- $\Theta_1 \subseteq [\sigma, \lambda]$ be such that $\sigma = \min \Theta_1$, $\max \Theta_1 = \lambda$,
- and let $\Theta_0 = [\sigma, \lambda] \backslash (\Theta_1 \cup \{\lambda\})$.

If $|\Theta_1| < \min \Theta_0$, then there is a ccc generic extension in which

$$\mathfrak{sp}(\mathfrak{i}) = \Theta_1 \cup \{\lambda\}.$$

Corollary (F., Shelah)

Assume GCH. Any countable set Θ of uncountable cardinals such that $\min \Theta$ is regular and $\sup \Theta = \max \Theta$ is of uncountable cofinality can be realized in a ccc generic extension as the spectrum of independence.

Corollary

Assume GCH and let $C \subseteq \{\aleph_n\}_{1 \leq n < \omega}$. Then there is a ccc generic extension in which

$$\mathfrak{sp}(\mathfrak{i}) = C.$$

Question:

Is it consistent that $\mathfrak{i} = \aleph_\omega$?

Thank you for your attention!